Journal of Statistical Physics, Vol. 105, Nos. 5/6, December 2001 (© 2001)

A Note on Eigenvalues of Liouvilleans

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Received June 21, 2001

Let L be the Liouvillean of an ergodic quantum dynamical system $(\mathfrak{M}, \tau, \omega)$. We give a new proof of the theorem of Jadczyk that eigenvalues of L are simple and form a subgroup of $\mathbb{R}$. If ω is a (τ, β)-KMS state for some $\beta \neq 0$ we show that this subgroup is trivial, namely that zero is the only eigenvalue of L. Hence, for KMS states ergodicity is equivalent to weak mixing.

KEY WORDS: Liouvillean ergodic theory; spectral theory; weak mixing; quantum statistical mechanics.

1. INTRODUCTION

Let $\mathfrak{M}$ be a von Neumann algebra on the Hilbert space $\mathscr{H}$ and $\mathfrak{M}_*$ its predual. The positive elements of $\mathfrak{M}_*$ satisfying $\omega(\mathbf{1}) = 1$ are called states. Let τ^t be a $\sigma(\mathfrak{M}_*, \mathfrak{M})$-continuous group of automorphisms of $\mathfrak{M}$. The pair $(\mathfrak{M}, \tau)$ is called a W^*-dynamical system. In the algebraic formalism of quantum statistical mechanics, the elements of $\mathfrak{M}$ describe observables of the physical system under consideration and the group τ specifies their time development.

A functional $\eta \in \mathfrak{M}_*$ is τ-invariant if $\eta \circ \tau^t = \eta$ for all t. A triple $(\mathfrak{M}, \tau, \omega)$, where ω is a τ-invariant state, is called quantum dynamical system. For our purposes we may assume without loss of generality that ω is a vector state, namely that $\omega(A) = (\Omega, A\Omega)$ for some $\Omega \in \mathscr{H}$, and that Ω is a cyclic and separating vector for $\mathfrak{M}$. In what follows, $(\mathfrak{M}, \tau, \omega)$ is a given quantum dynamical system satisfying these properties.

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0022-4715/01/1200-0937/0

The system $(\mathfrak{M}, \tau, \omega)$ is called ergodic if for all $A, B \in \mathfrak{M}$,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \omega(\tau^t(A)\, B)\, \mathrm{d}t = \omega(A)\, \omega(B),$$

and weak-mixing if

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\omega(\tau^t(A)\, B) - \omega(A)\, \omega(B)|^2\, \mathrm{d}t = 0.$$

Clearly, weak-mixing implies ergodicity.

It is known that ergodic properties of quantum dynamical systems can be characterized in spectral terms, in analogy with Koopman's lemma of classical ergodic theory.[1, 2] There exists a unique self-adjoint operator L on $\mathscr{H}$ such that for $A \in \mathfrak{M}$,

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL},$$

$$L\Omega = 0.$$

The operator L is a non-commutative analog of the classical Koopman operator. One can show (see Theorem 4.2 in ref. 2) that the quantum dynamical system $(\mathfrak{M}, \tau, \omega)$ is ergodic iff zero is a simple eigenvalue of L and weak-mixing iff L has no other eigenvalues except for a simple eigenvalue zero.

We denote by $\sigma_{\mathrm{p}}(A)$ the set of eigenvalues of a self-adjoint operator A. In this paper we give a new proof of the following result of Jadczyk[3] (see also [1, Theorem 4.3.27]).

Theorem 1.1. Assume that zero is a simple eigenvalue of L. Then all eigenvalues of L are simple and $\sigma_{\mathrm{p}}(L)$ is a subgroup of $\mathbb{R}$.

Remark. For dynamical systems which arise in classical ergodic theory this result goes back to Halmos and von Neumann, see refs. 4 and 5. The first results in the non-commutative case go back to ref. 6.

Our proof of Theorem 1.1 is somewhat simpler and perhaps more transparent then the argument in refs. 1 and 3. Moreover, the method of the proof yields some additional information. Let Δ be the modular operator associated to Ω, $\mathscr{L} = \log \Delta$ and $\sigma^t(A) = \mathrm{e}^{\mathrm{i}t\mathscr{L}} A \mathrm{e}^{-\mathrm{i}t\mathscr{L}}$ the group of modular automorphisms of $\mathfrak{M}$. Since $\mathscr{L}\Omega = 0$ the triple $(\mathfrak{M}, \sigma, \omega)$ is also a quantum dynamical system.

Theorem 1.2. Assume that zero is a simple eigenvalue of L and $\mathscr{L}$. Then

$$\sigma_{\mathrm{p}}(L) = \sigma_{\mathrm{p}}(\mathscr{L}) = \{0\}.$$

Let $\beta \neq 0$ be a real parameter. ω is called (τ, β)-KMS state if for all $A, B \in \mathfrak{M}$ there is a function $F_{A, B}$, analytic inside the strip $\{z \mid 0 < \operatorname{sign} \beta \operatorname{Im} z < |\beta|\}$, bounded and continuous on its closure, and satisfying the KMS-boundary condition

$$F_{A, B}(t) = \omega(A\tau^t(B)), \qquad F_{A, B}(t + \mathrm{i}\beta) = \omega(\tau^t(B)\, A).$$

A (τ, β)-KMS state describes a physical system in thermal equilibrium at inverse temperature β.

By a theorem of Takesaki,[1] ω is a (τ, β)-KMS state iff $\mathscr{L} = -\beta L$. Therefore, Theorem 1.2 implies the following somewhat surprising result.

Theorem 1.3. Assume that the system $(\mathfrak{M}, \tau, \omega)$ is ergodic and that ω is a (τ, β)-KMS state for some $\beta \neq 0$. Then the system is weak-mixing.

Theorems 1.2 and 1.3 show how modular structure associated to ω confines spectral structure of Liouvillean. Besides general interest, we expect that these theorems will be technically useful in the study of concrete models in quantum statistical mechanics. For an application of these theorems to the study of ergodic properties of Pauli–Fierz systems we refer the reader to ref. 7.

2. PROOFS

We assume that the reader is familiar with Tomita–Takesaki theory. For notational purposes we recall some basic results of this theory.

Let Δ and J be the modular operator and the modular conjugation associated to the vector Ω. For all $A \in \mathfrak{M}$, $J\Delta^{1/2}A\Omega = A^*\Omega$. Set $\mathscr{L} = \log \Delta$. Then

$$\begin{aligned} J\mathrm{e}^{\mathrm{i}tL} &= \mathrm{e}^{\mathrm{i}tL}J, \\ J\mathrm{e}^{\mathrm{i}t\mathscr{L}} &= \mathrm{e}^{\mathrm{i}t\mathscr{L}}J, \\ \mathrm{e}^{\mathrm{i}tL}\mathrm{e}^{\mathrm{i}s\mathscr{L}} &= \mathrm{e}^{\mathrm{i}s\mathscr{L}}\mathrm{e}^{\mathrm{i}tL}. \end{aligned} \tag{1}$$

By Tomita–Takesaki theorem, $J\mathfrak{M}J = \mathfrak{M}'$.

The natural cone $\mathscr{P}$ is the closure of the set $\{AJAJ\Omega \mid A \in \mathfrak{M}\} \subset \mathscr{H}$. The cone $\mathscr{P}$ is self-dual, namely $\mathscr{P} = \{\Psi \in \mathscr{H} \mid (\Psi, \Phi) \geqslant 0 \text{ for all } \Phi \in \mathscr{P}\}$. For every state $\eta \in \mathfrak{M}_*$, there is a unique vector $\Omega_\eta \in \mathscr{P}$ such that $\eta(A) = (\Omega_\eta, A\Omega_\eta)$. Moreover, the state η is τ-invariant iff $L\Omega_\eta = 0$. In particular, if zero is a simple eigenvalue of L, then ω is the unique τ-invariant state in $\mathfrak{M}_*$.

Proof of Theorem 1.1. Let E be an eigenvalue of L and Ω_E a (normalized) eigenvector associated to E. We show first that Ω_E is a cyclic and separating vector for $\mathfrak{M}$. Note that since $JL = -LJ$, $\Omega_{-E} := J\Omega_E$ is an eigenvector of L associated to the eigenvalue $-E$.

The states $\omega_{\pm E}(A) = (\Omega_{\pm E}, A\Omega_{\pm E})$ are τ-invariant, and hence for all $A \in \mathfrak{M}$,

$$(\Omega, A\Omega) = (\Omega_{\pm E}, A\Omega_{\pm E}). \tag{2}$$

Thus, if $A\Omega_E = 0$, then $A\Omega = 0$ and $A = 0$ (since Ω is separating). Hence Ω_E is separating. To prove that Ω_E is cyclic, let P' be the orthogonal projection on $\overline{\mathfrak{M}\Omega_E}$ and $Q' = \mathbf{1} - P'$. Then $Q' \in \mathfrak{M}'$ and $(\Omega_E, Q'\Omega_E) = 0$. Let $Q := JQ'J$. Then Q is an orthogonal projection, $Q \in \mathfrak{M}$, and

$$\begin{aligned} 0 &= (\Omega_{-E}, Q\Omega_{-E}) \\ &= (\Omega, Q\Omega) = \|Q\Omega\|^2. \end{aligned}$$

Since Ω is separating, $Q = 0$ and $P' = \mathbf{1}$. Hence Ω_E is cyclic.

Let U' be the linear map defined on $\mathfrak{M}\Omega_E$ by $U'A\Omega_E = A\Omega$. Since Ω_E is separating, the map U' is well-defined. Cyclicity of Ω_E and Eq. (2) yield that U' extends to a unitary map on $\mathscr{H}$. Since

$$U'AB\Omega_E = AB\Omega = AU'B\Omega_E,$$

$U' \in \mathfrak{M}'$.

Let L_E be the Liouvillean associated to Ω_E. Clearly, $L_E = L - E$ and

$$U'LU'^* = L_E = L - E. \tag{3}$$

Since zero is a simple eigenvalue of L, E is also a simple eigenvalue. Hence eigenvalues of L are simple. Note that if $U := JU'J$, then Eq. (3) implies

$$ULU^* = L + E. \tag{4}$$

Let now E_i, $i = 1, 2$ be two eigenvalues of L and let U_i', U_i be as in Eqs. (3) and (4). Set $W := U_2'U_1$. Then W is unitary and

$$WLW^* = L + E_1 - E_2.$$

It follows that $E_2 - E_1$ is an eigenvalue of L and $\sigma_{\mathrm{p}}(L)$ is a subgroup of $\mathbb{R}$. ∎

Proof of Theorem 1.2. By theorem 1.1 and the third relation in Eq. (1) it suffices to prove that $\sigma_{\rm p}(\mathscr{L}) = \{0\}$.

Let $E \in \sigma_{\rm p}(\mathscr{L})$, U', U be as in Eqs. (3) and (4), and $W = UU'$. W is unitary, $W\Omega \in \mathscr{P}$ and $W\mathscr{L} = \mathscr{L}W$. Since zero is a simple eigenvalue of $\mathscr{L}$ we must have $W\Omega = {\rm e}^{i\theta}\Omega$ for some real phase θ. Since $\mathscr{P}$ is a self-dual cone, $\theta = 0$ and

$$UJUJ\Omega = \Omega.$$

Using that $J\Omega = \Omega$ and $U \in \mathfrak{M}$ we derive

$$\begin{aligned} JU\Omega &= U^*\Omega = J\Delta^{1/2}U\Omega \\ &= J{\rm e}^{\mathscr{L}/2}U\Omega. \end{aligned}$$

It follows that $\mathscr{L}U\Omega = 0$, and since $\mathscr{L}U\Omega = -EU\Omega$, $E = 0$. Hence $\sigma_{\rm p}(\mathscr{L}) = \{0\}$. ∎

ACKNOWLEDGMENTS

We are grateful to Jan Dereziński for insisting on the question which has led to this note. Part of this work has been done during the visit of the first author to Aarhus University. V.J. is grateful to E. Skibsted and J. Dereziński for their hospitality. The research of the first author was partly supported by NSERC.

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